1. Sequences

Why Sequences

Here are Professor Bear's reasons to begin with a chapter about Sequences:

- Sequences are ordered lists that might have patterns. They are a great way to study patterns and only require a little knowledge of numbers and arithmetic to get started.
- Sequences are elementary functions. They map counting numbers 1, 2, 3, … to other numbers or items.
- Sequences are usually not given top billing in math class, yet they often appear on standardized tests.
- Sequences have a long and rich mathematical history. If you become a math major you will come back to them, and wonder why you didn't spend more time on them earlier. They are accessible to very young mathematicians.
- Monthly bank statements are sequences. Mortgage payments are sequences. Mortgage payments are calculated from geometric sequences. Drug treatments are sequences. Musical transcriptions are sequences. Sequences are pervasive.

Underpinnings of Sequences

To understand this material you need some experience with arithmetic and be able to do one and two digit addition, subtraction, multiplication and division.

It's also helpful but not necessary to know fractions.

Outcomes

After doing this chapter you'll have more experience with sequences and finding number patterns. You might find yourself motivated to learn basic number facts and fractions well enough to be able to play with sequences.

Solutions: Notes and Vocabulary

A **sequence** is an ordered list of things.

Here is a sequence of numbers: 1, 2, 3, 4, 5
Here is a sequence of letters: A, B, C, D
Here is a different sequence of numbers: 5, 4, 3, 2, 1

The "things" in the sequence are called **terms**. In the sequence 5, 4, 3, 2, 1

- first term is 5
- the third term is 3
- the fifth term is 1.

What is the first term of the sequence 1, 2, 3, 4, 5? 1

What is the fourth term of the sequence A, B, C, D? D

A sequence can be finite or infinite. The sequences above are all finite.

Here's another example of a finite sequence: 10, 9, 8

Here's another example: 400, 500, 600, 700, 800

Write down your own example of a finite sequence: ________________________
Have club members check one another's examples.

An infinite sequence is a list that goes on forever. It has an infinite number of terms.
We show that it goes on forever by three periods in a row at the end: . . .

Another name for "three periods in a row" is ellipsis.

Here is an example of an infinite sequence: 1, 2, 3, . . .

Another example of an infinite sequence: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\ \ldots$

Write down your own example of an infinite sequence: ________________________
Have club members check one another's examples.

A sequence of numbers might follow a pattern. If you know the pattern, you can figure out what the next term is. Can you write the next few terms of these sequences?

2, 4, 6, 8, 10, 12 , …

4, 7, 10, 13, 16, 19 , …

Some sequence patterns might be more challenging, such as this finite sequence:

3, 2, 1, 2, 3, 3, 3, 2, 2, 2, 3, 5, 5, … , 1

Can you fill in the missing terms? Think for a minute. A hint for this sequence is given later.

You can also use an ellipsis to write down a missing set of terms in a finite sequence.
For example: 1, 2, 3, … , 10
Is a short way to write 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
In the example 1, 2, 3, … , 10 the ellipsis stands for the missing terms 4, 5, 6, 7, 8, 9

In the following finite sequences, what are the missing terms?

Sequence	Missing Terms
10, 9, 8, … , 4	7, 6, 5
0, 3, 6, … , 24	9, 12, 15, 18, 21
77, 72, 67, … , 47	62, 57, 52

Hint for the sequence 3, 2, 1, 2, 3, 3, 3, 2, 2, 2, 3, 5, 5, … , 1 above: musical notes. The sequence corresponds to a melody. If you don't have musical experience, this example will not mean much to you.

The numbers correspond to the tune *Mary Had a Little Lamb*.

In that case, try to figure out the following pattern:

0, 3, 8, 15, 24, 35, 48, . . .

The pattern is to add progressively larger odd numbers.
0, 0+3, 3+5, 8+7, 15+9, and so on.

Solutions: Problems

1-1. Sample Sequences

Here are some sequences. What are the next three terms?

a. 5, 5.3, 5.6, 5.9, 6.2, . . . 6.5, 6.8, 7.

b. 2, 6, 18, 54, 162, . . . 486, 1458, 4374

c. 2, 5, 9, 14, 20, . . . 27, 35, 44

d. one, one thousand, one million, . . . one billion, one trillion, one quadrillion

1-2. Consecutive Terms

Consecutive terms are terms that exactly follow one another in a sequence.
In the sequence 1, 2, 3, … the terms 3, 4, 5, 6, and 7 are consecutive terms.
In the sequence A, B, C, … , Z the terms P and Q are consecutive terms.

Write three consecutive whole numbers: ______________, _____________ , ______________

Write three consecutive whole numbers bigger than a million:

____________________ , ___________________ , ____________________

Write three consecutive odd numbers: _____________, _____________ , _____________

Write three consecutive multiples of five: ____________ , _____________ , ___________

Write three consecutive prime numbers: ___________ , ____________ , _____________

Write three consecutive two-digit primes: __________ , ____________ , _____________

Write three consecutive perfect squares: ____________ , ____________ , ____________

Pose a more challenging problem involving consecutive numbers that you are pretty sure you are able to solve. Make it as different from the above problems as you can.

Example: Write three consecutive four-digit multiples of 17.

Pose an even more challenging problem involving consecutive numbers that you are pretty sure you are not able to solve (yet).

Example: What are the dates of the next three consecutive supernova explosions that will happen in the Milky Way?

Note to coach: In this problem, encourage club members to show one another their examples and check them to see if they are truly consecutive. This problem is good for group discussion.

1-3. Fibonacci Sequence

The Fibonacci sequence is: 1, 1, 2, 3, 5, 8, 13, . . .

a. The terms of the Fibonacci sequence follow a pattern. Describe the pattern.
   After the first two terms, each term is the sum of the previous two terms.

b. List the next three terms of the Fibonacci sequence. 21, 34, 55

c. The terms of the Fibonacci sequence are called Fibonacci numbers. How many prime Fibonacci numbers can you find?

   Fibonacci primes under 100 are: 2, 3, 5, 13, 89. The next one is 233.

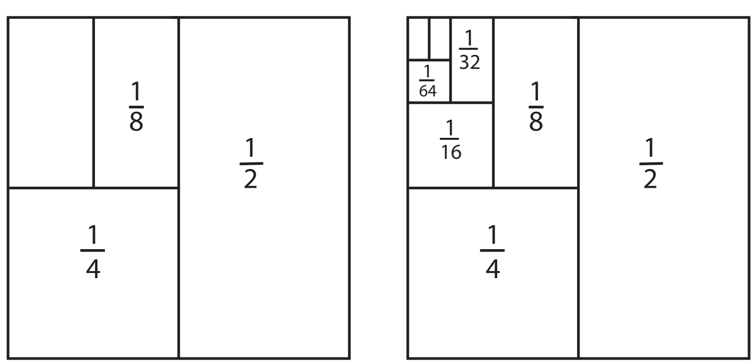
d. Choose any three consecutive terms of the Fibonacci sequence. Square the middle term and multiply the two other terms. Compare the results. Repeat this process with other groups of three consecutive terms. What is the pattern? (This is Cassini's Identity.)

   For any three consecutive terms, the square of the middle term has a difference of one from the product of the two outer terms.

1-4. Zeno's Paradox In A Box

The sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, . . .$

Is illustrated in this box:

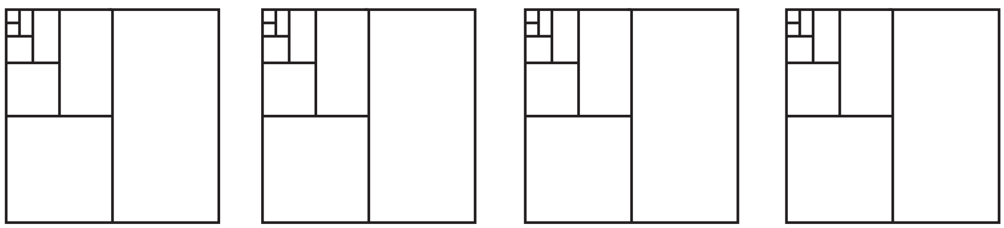


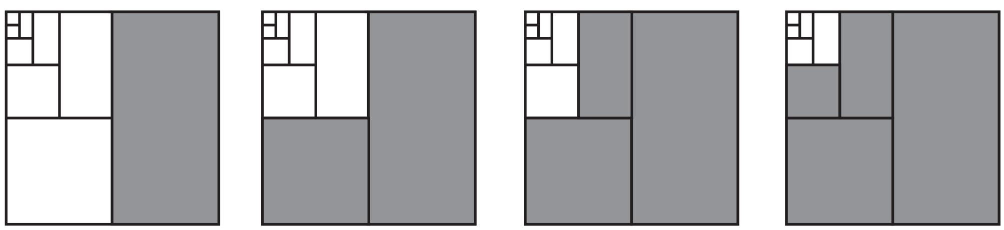
Draw in a few more terms in the box.

Describe what happens to the terms of the sequence, as they go on and on.
The terms get smaller and smaller.

Here is another sequence: $\frac{1}{2}, \; \frac{1}{2}+\frac{1}{4}, \; \frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \; . . .$

Illustrate this sequence in these boxes:





Describe what happens to the terms of this sequence, as they go on and on.
If you know how to add fractions, work out the terms and see if they follow a pattern.

The pattern is: ½, ¾, 7/8, 15/16, …
The terms of the sequence get closer and closer to 1. They fill up the box.

1-5. Days Of The Week

The days of the week form a sequence that repeats every seven terms.
a. What day of the week is today? ___________________

b. What day of the week will it be 100 days from now? ______________________
Because 98 is a multiple of 7, the day of the week will be the day 2 days later from the current day.

c. 500 days from now, will it be a different day of the week from today?
500 days from now, the day of the week will be three days later than the current day, because 497 is a multiple of seven.

d. Name a three-digit number for which you know that it will be the same day as today, that many days from now. 777 or any 3-digit multiple of 7.

There are several ways for people to understand this problem. One way is about remainders when you divide by 7. If the current day is Monday, then the 8^{th} day is Monday, and the 15^{th}, and so on—any number with a remainder of 1 when you divide by 7.

Another good strategy is to list the first several days and their numbers.

1-6. Counting Terms

This problem is about careful reading of what the question is asking, and careful reading of the sequence pattern.

a. How many terms are in the sequence: 1, 3, 5, 7, . . . , 993, 995, 997, 999 ? 500

b. Find the 1000^{th} term in the sequence: 2, 3, 4, 5, 6, . . . 1001

c. Find the 1000^{th} term in the sequence: 2, 4, 6, 8, . . . 2000

1-7. Cumulative Peanuts

Ella the elephant ate 1000 peanuts from Monday through Friday. Each day she ate 10 peanuts more than the previous day. How many peanuts did she eat on Friday?

Method 1: Write down how many peanuts Ella ate each day: Monday, X peanuts. Tuesday, X +10. Wednesday: X + 20. Thursday: X + 30. Friday: X + 40. Adding them all up, you get 5X + 100. So 5X is 900 and Ella ate 180 peanuts on Monday, which means she ate 220 peanuts on Friday.

Method 2: If she ate 1000 peanuts in 5 days, on average she ate 200 peanuts per day. Since there is an odd number of days, she eats the average on the middle day, Wednesday (before Wednesday, she eats less than average; after Wednesday, she eats more than average). If she eats 200 on Wednesday, she eats 220 on Friday.

1-8. Amoebas

An amoeba reproduces by dividing into two. Suppose it takes a day for an amoeba to eat enough food to be able to divide, and a scientist drops an amoeba into a pond in his back yard on January 1.

a. Write down the sequence of amoeba population values for January 1, January 2, and so on. 2, 4, 8, 16, …

b. The scientist notices that the pond is full of amoebas on February 10.
   On what date would the pond be full if he dropped two amoebas instead of one on January 1? February 9

1-9. Trina's Sequence Machine

Trina has a sequence machine. She gives it a first number and the machine manufactures the next term, according to these rules:

Rule 1: If the number is less than 10, add 3 to get the next term.
Rule 3: If the number is equal to 10, subtract 5 to get the next term.
Rule 4: If the number is greater than 10, subtract 6 to get the next term.

a. If the first term is 9, what is the 100^{th} term? 9
   The terms form a pattern: 9, 12, 6, 9, 12, 6, 9, 12, 6, …
   The 99^{th} term would be 6, and the 100^{th} term would be 9.

b. If the first term is 2, what is the 100^{th} term? 11
   The terms form a pattern: 2, 5, 8, 11, 5, 8, 11, 5, 8, 11, 5, 8, 11, …
   The 3^{rd}, 6^{th}, … 99^{th} terms are 8. So the 100^{th} term is 11.

1-10. Julio's Sequence Machine

Julio has a sequence machine. He gives it a first number and it manufactures the next terms according to these rules:

Rule 1: If the number is greater than 8, subtract 7 to get the next term.
Rule 2: If the number is less than or equal to 8, double it to get the next term.

Work out the sequences for various different starting numbers.

First Number	Julio's Sequence
0	0, 0, 0
1	1, 2, 4, 8, 16, 9, 2, 4, 8, 16, 9, 2, …
2	2, 4, 8, 16, 2, …
3	3, 6, 12, 5, 10, 3, …
4	4, 8, 16, 9, 2, 4, …
5	5, 10, 3, 6, 12, 5, …

6	6, 12, 5, 10, 3, 6, …
7	7, 14, 7, 14, 7, …
8	8, 16, 9, 2, 4, 8, …
9	9, 2, 4, 8, 16, 9, …

What patterns do you see?

Each sequence is different, because it has a different first term, but are some of Julio's sequences similar to each other, or related to each other?

What's a good way to define *similar sequence* in the world of Julio's sequences?

The sequences that start with 1, 2, 4, 8, and 9 have similar behavior. The sequences that start with 3, 5, and 6 are similar.

How To Deal With Sequence Problems

Learn the vocabulary:

- sequence
- consecutive
- term
- ellipsis

Look for patterns such as:

- adding
- subtracting
- multiplying
- perfect squares
- prime numbers
- odd/even numbers
- a steadily changing pattern
- use lists, charts, diagrams to help spot the pattern

There might be surprises such as:

- digits transposed
- sequence machines that have different patterns